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Light Fermion Finite Mass Effects in Non-relativistic Bound States

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Abstract

We present analytic expressions for the vacuum polarization effects due to a light fermion with finite mass in the binding energy and in the wave function at the origin of QED and (weak coupling) QCD non-relativistic bound states. Applications to exotic atoms, $\Upsilon(1s)$ and $t\bar{t}$ production near threshold are briefly discussed.

1 Introduction

It is well known that the vacuum polarization effects due to light fermions produce leading corrections to the Coulomb-like behavior of non-relativistic bound states both in QED [1] and in QCD [2]. If a light particle has a mass (m_l) of the order of the inverse Bohr radius, it can neither be approximated by a massless particle nor by a very heavy one. Hence any observable related to the bound state depends non-trivially on m_l. For a given bound state the m_l dependence in physical observables is usually calculated numerically [1,3–8] and only a few analytical results are available [9–12] [1]. In this letter we present further analytical formulas with the exact light fermion mass dependence for the leading corrections to the energy shift for arbitrary quantum numbers (n, l) and to the wave function at the origin for the ground state.

These formulas may be useful for quite a few physical systems of current interest. Any QED bound state built out of particles heavier than the electron may require them. For instance di-muonium, muonic hydrogen, pionium, pionic hydrogen and other simple hadronic atoms where the electron mass (m_e) is such that $m_e \sim \mu\alpha/n$, μ and n being the reduced mass and the principle quantum number respectively. It is worth mentioning that simple hadronic atoms are experiencing a renewed interest because they may allow to extract important information on the QCD scattering lengths for several isospin channels [13]. In particular the measurement of the decay width of pionium [14] at the 10% level will allow to extract a combination of scattering lengths with sufficient accuracy as to discern between the large and the small quark condensate scenario of QCD, namely between Chiral Perturbation Theory [15] and Generalized Chiral Perturbation Theory [16]. In fact, the leading corrections to the pionium decay width have been recently calculated in a systematic way using non-relativistic effective field theory techniques [9,17] (see [18], [5,19] and [20] for earlier relativistic, non-relativistic, and quantum mechanical calculations respectively). Fully analytic results have been obtained except for the contribution of the electron mass to the vacuum polarization where only a numerical result is available [5]. We shall fill this gap here and present the only remaining piece to have the full leading corrections to the pionium decay width in an analytic form.

For QCD non-relativistic bound states, $\Upsilon(1s)$ seems to be the only one amenable to a weak coupling analysis [21]. Using the results of [9], analytic expressions for the binding energy shift due to the finite charm mass have been recently presented in [22] [2]. We present here analytic results for these effects in the wave function at the origin.

An important non-relativistic weak coupling QCD system for the Next Linear Collider physics is the top quark-antiquark pair near threshold. The production cross-section has already been calculated at NNLO [24]. However, the calculations are done assuming the mass of the bottom and charm quarks zero. Our results may allow to include the leading

[1] We learnt about the three last references after completing our calculations.
[2] The extraction of the bottom $\overline{\mathrm{MS}}$ mass from the $\Upsilon(1s)$ mass also requires the bottom pole mass dependence on m_c [23].

effects of these masses.

The fact that our results can be applied to such a variety of a priory quite different physical systems can be easily understood in terms of modern effective field theories. Non-relativistic bound states have at least three dynamical scales: the hard scale (mass of the particles forming the bound state), the soft scale (typical relative momentum in the bound state) and the ultrasoft scale (typical binding energy in the bound state). Upon integrating out the hard scale local non-relativistic effective theories arise. These are NRQED for QED, NRQCD for QCD [25], and NRχL for the Chiral Lagrangian [3] [26]. Upon integrating out the soft scale (see [28,29] for the precise statements) effective theories which are local in time but non-local in space arise. The non-local terms in space are nothing but the usual quantum-mechanical potentials and only ultrasoft degrees of freedom [4] are left dynamical. The corresponding non-relativistic effective theories have been named pNRQED, pNRQCD and pNRχL for QED, QCD and the Chiral Lagrangian respectively [26,30], (the "p" stands for potential). Since the leading (mass independent) coupling of the photon field to the non-relativistic charged particles as well as the one of the gluon field to the non-relativistic quarks in the NR theories is universal, it produces the same potential in the pNR theories, and hence they all can be discussed at once. If there is a light (relativistic) charged particle in QED or a light (relativistic) quark in QCD whose mass is of the order of the soft scale, it must be integrated out keeping the mass dependence exact, which produces a light fermion mass dependent correction to the static potential.

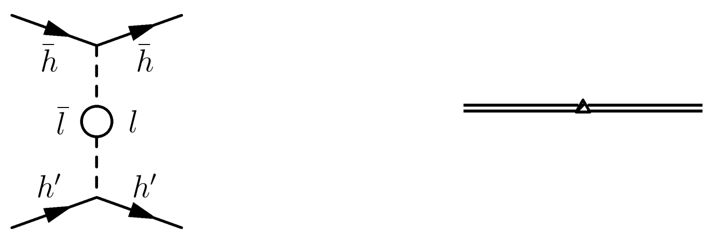

Fig.1: Matching between the non-relativistic theory and the potential one.

When matching the NR theories to the pNR theories only the diagram of Fig. 1 gives rise to a potential which contributes to the leading effect. For QED (on-shell scheme) it reads,

$$V_{vpc}(|\mathbf{x}|) = -\frac{\alpha}{\pi}\frac{\alpha}{|\mathbf{x}|}\int_0^1 dv\, \frac{v^2\left(1-\frac{v^2}{3}\right)}{(1-v^2)} e^{-\frac{m_l|\mathbf{x}|}{\sqrt{1-v^2}}} \qquad (1)$$

and for QCD ($\overline{\mathrm{MS}}$):

[3] We apologize to the authors of ref. [5] for slightly modifying their previously given name, namely *Non Relativistic Chiral Perturbation Theory.* The reason is that the calculation in the non-relativistic regime cannot be organized according to the chiral counting any longer [9].

[4] In the language of the threshold expansions in QCD [27] these correspond to ultrasoft gluons and potential quarks.

$$
\begin{aligned}
V_{vpc}(|\mathbf{x}|) &= -\frac{C_F T_F \alpha_s}{\pi}\frac{\alpha_s}{|\mathbf{x}|}\left\{\int_0^1 dv\, \frac{v^2\left(1-\frac{v^2}{3}\right)}{(1-v^2)} e^{-\frac{m_l|\mathbf{x}|}{\sqrt{1-v^2}}} + \frac{1}{3}\log\left(\frac{m_l^2}{\nu^2}\right)\right\} \\
C_F &= \frac{N_c^2-1}{2N_c} = \frac{4}{3} \quad , \quad T_F = \frac{1}{2}
\end{aligned}
\tag{2}
$$

If N_f is the number of flavors lighter that m_l, the $\alpha_s(\nu)$ above runs with N_f+1 flavors. Notice that the difference between the QED and the QCD case is, apart from the trivial color factors $\alpha/|\mathbf{x}| \to C_F\alpha_s/|\mathbf{x}|$ and $\alpha/\pi \to T_F\alpha_s/\pi$, a term which can be absorbed in a redefinition of the Coulomb potential [31]. Hence for the actual calculation we shall only deal with (1) and use these facts to extend our results to the QCD realm.

2 Energy Shift

For the energy shift we obtain ($\xi := \frac{nm_l}{\mu\alpha}$):

$$
\begin{aligned}
\delta E_{nl}(\xi) = -\frac{2\alpha}{3\pi}E_n\Bigg\{&\frac{5}{3} - \frac{3\pi}{2}n\xi + (n(2n+1)+(n+l)(n-l-1))\,\xi^2 - \\
&-\pi n\left(\frac{1}{3}(n+1)(2n+1)+(n+l)(n-l-1)\right)\xi^3 - \\
&-\frac{1}{(2n-1)!}\sum_{k=0}^{n-l-1}\begin{pmatrix} n-l-1 \\ k \end{pmatrix}\begin{pmatrix} n+l \\ 2l+1+k \end{pmatrix}\xi^{2(n-l-1-k)} \\
&\frac{d^{2n-1}}{d\xi^{2n-1}}\left[\xi^{2(k+l)+1}(2-\xi^2-\xi^4)F_1(\xi)\right]\Bigg\}
\end{aligned}
\tag{3}
$$

$$
F_1(\xi) := \begin{cases} \frac{1}{\sqrt{\xi^2-1}}\arccos\frac{1}{\xi} & \text{if } \xi > 1, \\ 1 & \text{if } \xi = 1, \\ \frac{1}{\sqrt{1-\xi^2}}\log\left[\frac{1+\sqrt{1-\xi^2}}{\xi}\right] & \text{if } \xi < 1. \end{cases}
$$

where $E_n = -\mu\alpha^2/2n^2$ is the Coulomb energy. For ξ large, namely $m_l >> \mu\alpha/n$, it reduces to

$$
\begin{aligned}
\delta E_{nl}(\xi \to \infty) \to \frac{2\alpha}{3\pi}\frac{E_n}{\xi^{2l+2}}\Bigg\{&\frac{(n+l)!}{(n-l-1)!(2l+1)!}\frac{(2l)!!}{(2l+1)!!} \\
&\left(2-\frac{(2l+2)}{(2l+3)}-\frac{(2l+2)(2l+4)}{(2l+3)(2l+5)}+O\left(\frac{1}{\xi}\right)\right)\Bigg\},
\end{aligned}
\tag{4}
$$

whereas for ξ small, namely $m_l << \mu\alpha/n$, we obtain

$$\begin{aligned} \delta E_{nl}(\xi \to 0) \to & -\frac{2\alpha}{3\pi}E_n \left\{ \frac{5}{3} + 2\left(\psi(n+l+1) - \psi(1)\right) - 2\log\frac{2}{\xi} - \right. \\ & -\frac{3\pi}{2}n\xi + \frac{3}{2}\left(n(2n+1) + (n+l)(n-l-1)\right)\xi^2 - \\ & \left. -\pi n \left(\frac{1}{3}(n+1)(2n+1) + (n+l)(n-l-1)\right)\xi^3 + O(\xi^4) \right\} \end{aligned} \tag{5}$$

where we have used (13). The key steps to obtain (3) are given in the Appendix B. We have done the following checks. For the 1S state (3) reduces to the formula (5.3) of ref. [9]. The energy shifts for the 1S, 2S, 2P, 3S, 3P and 3D states agree with the early analytical formulas of ref. [10]. For ξ large, we reproduce the well-known positronium like limit for $l = 0$ (to be precise we agree with the correction to the energy obtained using formula (2.8) of [28]). We also agree for $l = 1$ with formula (32) of ref. [4]. For ξ small, we can compare with known results for massless quarks in QCD. For arbitrary n and l we agree with formula (13) of ref. [31]. For $l = n - 1$ we agree with $O(\xi^0)$ and $O(\xi^1)$ of formula (14) in ref. [12] [5] but disagree with their $O(\xi^2)$ result (the $O(\xi^3)$ is not displayed in [12]). Notice that for ξ large enormous cancellations occur in formula (3) and hence the analytic expansion (4) may prove very useful.

3 Wave Function at the Origin

The correction for the wave function at the origin for $n = 1$ states reads

$$\begin{aligned} \delta\Psi_{10}(\mathbf{0}) = & -\frac{\alpha}{\pi}\Psi_{10}(\mathbf{0}) \left[\left\{ \frac{5}{9} - \frac{\pi}{4}\xi + \frac{1}{3}\xi^2 - \frac{\pi}{6}\xi^3 + \frac{1}{3}(\xi^4 + \xi^2 - 2)F_1(\xi) \right\} + \right. \\ & \left\{ \frac{11}{18} - \frac{2}{3}\xi^2 + \frac{2\pi}{3}\xi^3 - \frac{1}{6}(12\xi^4 + \xi^2 + 2)F_1(\xi) - \frac{1}{6}\frac{(4\xi^4 + \xi^2 - 2)}{(\xi^2 - 1)}(1 - \xi^2 F_1(\xi)) \right\} + \\ & + \left\{ \frac{2}{3} + \frac{\pi}{4}\xi - \frac{1}{9}\xi^2 + \frac{13\pi}{18}\xi^3 - \frac{1}{9}(13\xi^4 - 11\xi^2 - 11)F_1(\xi) - \right. \\ & \left. \left. -\frac{1}{3}(4\xi^3 + 3\xi)F_2(\xi) + \frac{1}{3}(4\xi^4 + \xi^2 - 2)F_3(\xi) + \frac{1}{3}\left(4\xi^2 + \frac{11}{3}\right)\log\frac{\xi}{2} \right\} \right] \end{aligned} \tag{6}$$

where $\Psi_{10}(\mathbf{x})$ is the Coulomb wave function. The first bracket corresponds to the zero photon exchange and has already been calculated analytically in [5]. The second and third brackets correspond to the pole subtraction and multi-photon exchange contributions respectively. $F_i(\xi)$, $i = 2, 3$ are defined as follows:

[5] Taking $\epsilon_n = 0$ in that reference and upon correcting an obvious misprint $\kappa_1 \to \kappa_n$.

$$
F_2(\xi) = \int_0^{\frac{\pi}{2}} d\theta \, \log\left[\frac{\sin\theta + \xi}{\sin\theta}\right]
$$
$$
F_3(\xi) = \int_0^{\frac{\pi}{2}} d\theta \, \frac{1}{\sin\theta + \xi} \log\left[\frac{\sin\theta + \xi}{\sin\theta}\right] \tag{7}
$$

$F_2(\xi)$ and $F_3(\xi)$ can be expressed in terms of Clausen integrals and dilogarithms. We present the explicit formulas in Appendix A. The key steps in order to obtain (6) are given in Appendix B.

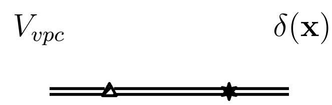

Fig.2: Diagram rendering the correction to the wave function at the origin. The double line is the Coulomb propagator of the non-relativistic pair and the star a local ($\delta(\mathbf{x})$) potential.

For ξ large, namely $m_l >> \mu\alpha$, (6) behaves like

$$
\delta\Psi_{10}(\mathbf{0})_{\xi\to\infty} \to \frac{\alpha}{\pi}\Psi_{10}(\mathbf{0})\left[\frac{3\pi}{16\xi} + \frac{107}{225\xi^2} + \frac{4}{15\xi^2}\log\frac{\xi}{2} + O\left(\frac{1}{\xi^3}\right)\right] \tag{8}
$$

This result must be compatible with the one obtained by integrating out the light fermion first and then calculating the electromagnetic potential. In the case $m >> m_l >> \mu\alpha/n$ (for simplicity, we are assuming $h = h'$, m being the mass of the non-relativistic particles) we expect that a local non-relativistic effective theory is obtained after integrating out the energy scale m_l and the associated three momentum scale $\sqrt{mm_l}$ for the non-relativistic particle. The leading term in (8) corresponds to the contribution that would be obtained from the local term induced by the diagram in Fig. 3. The logarithm in the subleading term corresponds to the iteration of two delta function potentials in quantum mechanics (see formula (5.5) in [9]). The second delta function is due to the contribution to the electromagnetic potential of the dimension six photon operator (see [32]) which arises after integrating out a heavy particle [33].

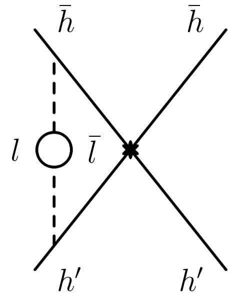

Fig.3: Vacuum polarization correction to the decay width at leading order in $\frac{1}{\xi}$ when $\xi \to \infty$. Energies and momenta of order m_l and $\sqrt{m m_l}$ respectively dominate the graph and hence the Coulomb resummation leads to subleading effects.

For ξ small, namely $m_l << \mu\alpha$, we obtain

$$\delta\Psi_{10}(\mathbf{0})(\xi \to 0) \to -\frac{\alpha}{\pi}\Psi_{10}(\mathbf{0})\left[\frac{3}{2} - \frac{\pi^2}{9} - \log\frac{2}{\xi} - \frac{3}{2}\xi^2 + O(\xi^3)\right] \tag{9}$$

We have made the following checks. For ξ large and small, the leading term of (8) and (9) agree with formulas (22) and (23) of ref. [4] respectively (the next-to-leading terms are not displayed in [4]). For ξ small we can also compare with known results for massless quarks in QCD. We agree with the $O(\alpha_s)$ correction of formula (69) of ref. [35]. We have also checked that the formula (6) reproduces the numerical results obtained for di-muonium and pionium in refs. [11] and [5] respectively, and we also agree numerically with the analytical result in terms of a non-trivial integral of ref. [4].

4 Applications

4.1 Exotic Atoms

We have listed in Table I and Table II the corrections to some energy splittings and to the wave function at the origin respectively of simple exotic atoms of current interest. This purely electromagnetic corrections must be conveniently taken into account if one wants to obtain precise information of the strong scattering lengths from hadronic atoms.

TABLE I

	$\frac{\delta E_{21}-\delta E_{10}}{\alpha(E_2-E_1)}$	$\frac{\delta E_{31}-\delta E_{10}}{\alpha(E_3-E_1)}$	$\frac{\delta E_{21}-\delta E_{20}}{\alpha E_2}$
pK^-	.44593	.38629	-.10453
$p\pi^-$	.18103	.15388	-.056337
$p\mu^-$	.13616	.11548	-.044443

Vacuum polarization induced energy splittings for some exotic atoms.

TABLE II

	$\xi = \frac{m_e}{\mu\alpha}$	$\frac{\delta_{zph}\Psi(\mathbf{0})}{\alpha\Psi(\mathbf{0})}$	$\frac{\delta_{ps}\Psi(\mathbf{0})}{\alpha\Psi(\mathbf{0})}$	$\frac{\delta_{mph}\Psi(\mathbf{0})}{\alpha\Psi(\mathbf{0})}$	$\frac{\delta\Psi(\mathbf{0})}{\alpha\Psi(\mathbf{0})}$
K^-p	.21648	.34290	.15454	.09650	.59394
K^+K^-	.28369	.29837	.12958	.08785	.51581
π^-p	.57635	.19613	.07285	.06166	.33064
$K^+\pi^-$	.64357	.18237	.06549	.05741	.30527
μ^-p	.73738	.16627	.05703	.05222	.27552
$\pi^+\pi^-$	1.00344	.13338	.04052	.04099	.21490
$\mu^+\mu^-$	1.32550	.10793	.02876	.03184	.16853

Vacuum polarization correction to the ground state wave function at the origin of some exotic atoms.

4.2 $\Upsilon(1s)$ *and* $t\bar{t}$

The current calculations of heavy quarks near threshold assume that the remaining lighter quarks are massless. This approximation is difficult to justify *a priori* at least in two cases. For the $\Upsilon(1s)$ system the typical relative momentum $m_b\alpha_s/2 \sim 1.3$ GeV. [34] is of the same order as the charm mass $m_c \sim 1.5$ GeV. The effects of a finite charm mass in the binding energy have been recently quantified in [22]. We give in Table III the size of these effects in the wave function at the origin. For the $t\bar{t}$ production near threshold at a relative momentum $m_t\alpha_s/2 \sim 18$ GeV. the effects of a finite bottom mass $m_b \sim 5$ GeV. should be noticeable. In order to estimate them, we also show in Table III the size of this effect, both for bottom and charm, in the wave function at the origin for the would-be-toponium $(1s)$ state.

TABLE III

	$\xi = \frac{m_c}{C_F\alpha_s\mu}$	$\xi = \frac{m_b}{C_F\alpha_s\mu}$	$\frac{\delta\Psi(\mathbf{0})_{\xi\neq 0}-\delta\Psi(\mathbf{0})_{\xi=0}}{\alpha_s\Psi(\mathbf{0})}$
$\bar{b}b$	1.4		.088
$\bar{t}t$		.28	.011
$\bar{t}t$	.10		.0019

Vacuum polarization correction to wave function at the origin in quarkonia. $\overline{\mathrm{MS}}$ has been used.

If the corrections were organized in a series of α_s/π multiplied by numbers of order 1, one may conclude that the leading effects of a finite quark mass are: (i) in the $\Upsilon(1s)$ system for charm more important than the next to leading corrections [35]; (ii) in the $t\bar{t}$ system near threshold for bottom (charm) as important as (less important than) the next to leading corrections [24]. However, the relativistic corrections do not have the π suppression and some radiative corrections are enhanced by factors of β_0. In practise the next to leading corrections are comparable to the leading ones, even for the $t\bar{t}$ system (see [36] for a discussion). This makes the actual size of the finite mass effects smaller than the next to leading order corrections in all the cases above.

Acknowledgements

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Appendix A

$F_2(x)$ in (7) can be expressed in terms of Clausen integrals. We get

$$F_2(x) = \begin{cases} 2Cl_2(\arcsin\, x) - \frac{1}{2}Cl_2(2\arcsin\, x) & \text{if } x < 1 \\ 2\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \sim 1.831932 & \text{if } x = 1 \\ \begin{array}{c} -i\, Li_2(-x-\sqrt{x^2-1}) + i\, Li_2(i\left(-x+\sqrt{x^2-1}\right))- \\ -i\, Li_2(-x+\sqrt{x^2-1}) + i\, Li_2(-i\left(x+\sqrt{x^2-1}\right))+ \\ +\frac{\pi}{8}\left(i\pi + 4\log(2x) - 4\log(1+ix-i\sqrt{x^2-1}) - 4\log(1+ix+i\sqrt{x^2-1})\right)+ \\ +2\left(\frac{\pi}{2} - i\log\left(\sqrt{\frac{x+1}{2}} - \sqrt{\frac{x-1}{2}}\right)\right)\left(-2\,i\ \arctan(\frac{x-1}{\sqrt{x^2-1}})+\right. \\ +\log(1+x-\sqrt{x^2-1}) - \log(1+i\,x-i\,\sqrt{x^2-1})+ \\ \left. +\log(1+i\,x+i\,\sqrt{x^2-1}) - \log(1+x+\sqrt{x^2-1})\right) \end{array} & \text{if } x > 1. \end{cases} \tag{10}$$

Recall that the Clausen integral is defined as

$$Cl_2(x) := -\int_0^x d\theta\, \log\left(2\sin\frac{\theta}{2}\right) = i\frac{\pi^2}{6} - \frac{i}{4}x^2 - x\log(ie^{-i\frac{x}{2}}) - i\,Li_2(e^{i\,x}) \tag{11}$$

$F_3(x)$ in (7) can be expressed in terms of dilogarithms. We get

$$
\begin{aligned}
F_3(x) = \frac{1}{\sqrt{1-x^2}} \Bigg[& Li_2\left(-\frac{(1+a+b)}{2b}\right) - Li_2\left(-\frac{(1+a-b)}{2b}\right) + Li_2\left(-\frac{2b}{(1+a+b)}\right) - \\
& -Li_2\left(\frac{2b}{(1+a+b)}\right) - Li_2\left(-\frac{(a+b)}{2b}\right) + Li_2\left(-\frac{(a-b)}{2b}\right) - Li_2\left(-\frac{2b}{(a+b)}\right) + \\
& +Li_2\left(\frac{2b}{(a+b)}\right) + Li_2\left(\frac{(a+b)}{(1+a+b)}\right) - Li_2\left(\frac{(a-b)}{(1+a-b)}\right) + \\
& + \log(bx)\, \log\left(\frac{(1+a+b)}{(1+a-b)}\right) \Bigg] \qquad \text{if} \quad x < 1,
\end{aligned}
$$

where $a := x^{-1}$ and $b := \frac{\sqrt{1-x^2}}{x}$.

$$F_3(1) = 2 - \log 2$$

$$
\begin{aligned}
F_3(x) = \frac{1}{i\sqrt{x^2-1}} \Bigg[& Li_2\left(-\frac{(1+a+ib)}{2ib}\right) - Li_2\left(-\frac{(1+a-ib)}{2ib}\right) + Li_2\left(-\frac{2ib}{(1+a+ib)}\right) - \\
& -Li_2\left(\frac{2ib}{(1+a+ib)}\right) - Li_2\left(-\frac{(a+ib)}{2ib}\right) + Li_2\left(-\frac{(a-ib)}{2ib}\right) - Li_2\left(-\frac{2ib}{(a+ib)}\right) + \\
& +Li_2\left(\frac{2ib}{(a+ib)}\right) + Li_2\left(\frac{(a+ib)}{(1+a+ib)}\right) - Li_2\left(\frac{(a-ib)}{(1+a-ib)}\right) + \\
& +2\arctan\left(\frac{b}{1+a}\right)\left(i\log(bx) - \frac{\pi}{2}\right) \Bigg] \qquad \text{if} \quad x > 1, \qquad (12)
\end{aligned}
$$

where $a := x^{-1}$ and $b := \frac{\sqrt{x^2-1}}{x}$.

In order to make contact with the expressions found in the literature for the massless limit of (3) the following formula is useful

$$\psi(2n) - \frac{(n+l)!(n-l-1)!}{(2n-1)!} \sum_{k=0}^{n-l-2} \frac{(2(n-l-1-k)-1)!(2(k+l)+1)!}{(n-l-k-1)!^2(2l+1+k)!k!} = \psi(n+l+1) \qquad (13)$$

Appendix B

We sketch here the main steps which lead to our analytic formulas. For the energy shift we have to calculate ($v = \sqrt{1-x^2}$)

$$\delta E_{nl} =< nl|V_{vpc}|nl > = \frac{2\alpha E_n}{3\pi} \frac{(n+l)!}{(n-l-1)!(2l+1)!}\, \xi^{2n-2l-2} \int_0^1 dx \frac{x^{2l+1}}{(x+\xi)^{2n}} \sqrt{1-x^2}\left(2+x^2\right)$$

$$F(-(n-l-1),-(n-l-1),2l+2;\frac{x^2}{\xi^2}). \tag{14}$$

Since $l<n$ the hypergeometric function above reduces to a polynomial

$$F(-(n-l-1),-(n-l-1),2l+2;z)=\sum_{j=0}^{n-l-1}\frac{(2l+1)!(n-l-1)!^2}{(n-l-1-j)!^2(j+2l+1)!j!}z^j. \tag{15}$$

Hence

$$\delta E_{nl}=\frac{2\alpha E_n}{3\pi}\sum_{j=0}^{n-l-1}\left(\begin{array}{c}n-l-1\\ j\end{array}\right)\left(\begin{array}{c}n+l\\ n-l-j-1\end{array}\right)\xi^{2n-2l-2j-2}$$
$$\times\int_0^1 dx\,\frac{x^{2l+2j+1}}{(\xi+x)^{2n}}\sqrt{1-x^2}\left(2+x^2\right). \tag{16}$$

and upon writing

$$\left(\frac{1}{\xi+x}\right)^{2n}=-\frac{d^{2n-1}}{d\xi^{2n-1}}\frac{1}{\xi+x} \tag{17}$$

and making the change $x\to\sin\theta$ we obtain (3).

For the wave function at the origin we have to calculate:

$$\delta\Psi_{n0}(\mathbf{0})\Psi_{n0}(\mathbf{0})=\lim_{E\to E_n}<n0|\delta(\mathbf{x})\left(\frac{1}{E-H}-\frac{|n0><n0|}{E-E_n}\right)V_{vpc}|n0> \tag{18}$$

Upon using the following representation for the Coulomb propagator [37]

$$<\mathbf{x}|\frac{1}{E-H}|\mathbf{y}>=\sum_{l=0}^{\infty}G_l(x,y,E)\sum_{m=-l}^{l}Y_l^m\left(\frac{\mathbf{x}}{x}\right)Y_l^{*m}\left(\frac{\mathbf{y}}{y}\right) \tag{19}$$

$$G_l(x,y,E)=-4k\mu(2kx)^l(2ky)^le^{-k(x+y)}\sum_{n'=1}^{\infty}\frac{L_{n'-1}^{2l+1}(2kx)L_{n'-1}^{2l+1}(2ky)\Gamma(n')}{(n'+l-\frac{\mu\alpha}{k})\Gamma(n'+2l+1)} \tag{20}$$

where $k^2=-2\mu E$, (18) can be split into three pieces

$$\delta\Psi_{n0}(\mathbf{0})=\delta_{ps}\Psi_{n0}(\mathbf{0})+\delta_{zph}\Psi_{n0}(\mathbf{0})+\delta_{mph}\Psi_{n0}(\mathbf{0}) \tag{21}$$

The first piece (pole subtraction) corresponds to the term $n'=n$ in the sum (20) and it can be calculated using the formulas given above. The remaining pieces read

$$\delta_{zph}\Psi_{n0}(\mathbf{0})+\delta_{mph}\Psi_{n0}(\mathbf{0})=\frac{\alpha}{\pi}\Psi_{n0}(\mathbf{0})\int_0^1 dv\,\frac{v^2\left(1-\frac{v^2}{3}\right)\xi^{n-1}}{(\xi+\sqrt{1-v^2})^{n+1}}\sum_{n'=1,n'\neq n}^{\infty}\frac{n'n}{n'-n}$$
$$\left(\frac{\xi}{\xi+\sqrt{1-v^2}}\right)^{n'-1}F\left(-(n-1),-(n'-1);2;\frac{1-v^2}{\xi^2}\right). \quad (22)$$

Again the hypergeometric function above reduces to a polynomial. For $n=1$ it reduces in fact to 1 and the sum over n' can be carried out explicitly. We obtain:

$$\delta_{zph}\Psi_{10}(\mathbf{0})+\delta_{mph}\Psi_{10}(\mathbf{0})=\frac{\alpha}{\pi}\Psi_{10}(\mathbf{0})\int_0^1 dv\frac{v^2\left(1-\frac{v^2}{3}\right)}{(\xi+\sqrt{1-v^2})^2}\left\{\frac{\xi}{\sqrt{1-v^2}}+\log\left(\frac{\xi+\sqrt{1-v^2}}{\sqrt{1-v^2}}\right)\right\} \quad (23)$$

where the first term corresponds to the zero photon exchange and the second one to the multiphoton exchange. Again the change of variable $v\rightarrow\cos\theta$ and a number of manipulations allow us to obtain (6) from the above.

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